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The Variance of Information Loss as a Characteristic Quantity of Dynamical Chaos

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The cumulants of the information loss are discussed as characteristic measures of dynamical chaos. They are extensions of the Liapunov exponent and Kolmogorov entropy, which are given by mean values of the information loss. The most important cumulant of higher than first order is the variance. It is discussed in particular for the logistic map.

KEY WORDS: Characterization of chaos; logistic map.

1. VARIANCE AND OTHER CUMULANTS OF INFORMATION LOSS

In the theory of chaos it is a central question to seek for quantities that characterize individual features of special types of dynamical chaos. Well-known quantities of this kind are the "invariant" or "natural" measure, the Liapunov exponent or Kolmogorov entropy, and the correlation measures, described, for instance, in Ref. 1–14. Further characteristic quantities have been recently proposed by Grassberger and Procaccia.[15–17]

In the following a class of measures will be introduced that are independent of the already known measures and thus describe new characteristic properties of dynamical chaos.

Dynamical Map. For a given dynamical "map"

$$\mathbf{x}^{(n+1)} = \mathbf{f}(\mathbf{x}^{(n)}) \tag{1.1}$$

in a d-dimensional parameter space

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \tag{1.2}$$

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the loss of information in one interaction step from n to $n+1$, expressed by the number of bits multiplied by log 2, is

$$\Delta I^{(n)} = \log |A(\mathbf{x}^{(n)})| \tag{1.3}$$

(compare, e.g., Ref. 2), where A is the Jacobian

$$A(\mathbf{x}) = \operatorname{Det} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \tag{1.4}$$

The invariant measure[13] of a map f is

$$\rho(\mathbf{x}) = \lim_{N \to \infty} N^{-1} \sum_{n=0}^{N-1} \delta(\mathbf{x} - \mathbf{x}^{(n)}) \tag{1.5}$$

It is a density in the d-dimensional $\mathbf{x}$ space. With it any mean value of a function $a(\mathbf{x})$ over the mapping process

$$\langle a \rangle = \lim_{N \to \infty} N^{-1} \sum_{n=0}^{N-1} a(\mathbf{x}^{(n)}) \tag{1.6}$$

if the limit exists can be written as the space integral

$$\langle a \rangle = \int dx\, \rho(\mathbf{x})\, a(\mathbf{x}) \tag{1.7}$$

Dynamical Flow. A dynamical "flow" in the $\mathbf{x}$ space is given by a set of autonomic differential equations in time t:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1.8}$$

It can be conceived as a map with infinitesimally small steps obtained by a limiting process $\tau \to 0$ with

$$(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)})/\tau \to \dot{\mathbf{x}} \tag{1.9}$$

$$[\mathbf{f}(\mathbf{x}) - \mathbf{x}]/\tau \to \mathbf{F}(\mathbf{x}) \tag{1.10}$$

$$|A| \to 1 + \tau \operatorname{div} \mathbf{F} \tag{1.11}$$

$$\Delta I/\tau \to \dot{I} = \operatorname{div} \mathbf{F} \tag{1.12}$$

The invariant measure (1.5) goes over into

$$\rho(\mathbf{x}) = \lim_{T \to \infty} T^{-1} \int_0^T dt\, \delta(\mathbf{x} - \mathbf{x}(t)) \tag{1.13}$$

where $\mathbf{x}(t)$ is the solution of the differential equation (1.8) with given initial values $\mathbf{x}(0)$. Any mean value of a function $a(\mathbf{x})$ over the process $\mathbf{x}(t)$

$$\langle a \rangle = \lim_{T \to \infty} T^{-1} \int_0^T dt\, a(\mathbf{x}(t)) \tag{1.14}$$

can be written as the space integral (1.7).

Information Loss Cumulants. In case of a map the information loss during one iteration step is ΔI; in case of a flow the information loss per unit of time is $\dot{I}$. In the following we shall denote both quantities uniformly by J and call them "information loss" or "simply loss." Thus, for the map,

$$J = \log |A| \tag{1.15}$$

and for the flow,

$$J = \operatorname{div} \mathbf{F} \tag{1.16}$$

The cumulants C_k of the loss J are defined by the power expansion with respect to a parameter η:

$$\log\langle \exp(\eta^J) \rangle = \sum_{k=0}^{\infty} C_k \eta^k / k! \tag{1.17}$$

k is called the "order" of the cumulant C_k. These loss cumulants are independent of each other and characterize independent properties of the dynamics (map or flow) in the same way as moments of different order of a random quantity characterize independent properties of a probability distribution. In particular, they can characterize dynamical chaos. Unlike the moments $\langle J^k \rangle$, the cumulants C_k have the distinguishing property of being additive for uncoupled dynamics occurring independently in separate subspaces of $\mathbf{x}$, whereas the moments of order $k > 1$ mix the subspaces. It is easily seen that the left-hand side of Eq. (1.7), the "generating function" of the cumulants, already possesses this distinguishing property of the cumulants. This function is, up to a factor, equal to the Rényi information,[18] which in connection with chaos has been discussed by Grassberger and Procaccia.[15–17]

The first cumulant of J is the mean value

$$C_1 = \langle J \rangle \tag{1.18}$$

In the one-dimensional case $d = 1$ it is the Liapunov exponent λ and is equal to the Kolmogorov entropy.[11,19–25]

The second cumulant is the variance of the information loss

$$C_2 = V^2 = \langle J^2 \rangle - \langle J \rangle^2 \tag{1.19}$$

It is a new characteristic quantity of chaos and generally of a map or flow, and shall be discussed in detail in the following. It is a measure for the inhomogeneity of the loss J in $\mathbf{x}$ space. It vanishes for a totally homogeneous loss J and becomes positive and larger the more the loss varies in space.

We make the following remark. For dimensionality $d > 1$ the Kolmogorov entropy is not defined by $\langle J \rangle$, but by the sum of all positive $\langle J_s \rangle$, where, in case of the map,

$$J_s = \log \left| \frac{\partial f_s}{\partial x_s} \right| \tag{1.20}$$

and in case of the flow,

$$J_s = \frac{\partial F_s}{\partial x_s} \tag{1.21}$$

($s = 1, 2, \ldots, d$).[24] Correspondingly, a loss variance could be introduced by an alternative definition,

$$V_s^2 = \langle J_s^2 \rangle - \langle J_s \rangle^2 \tag{1.22}$$

for any direction x_s in $\mathbf{x}$ space. The same could be done for the cumulants of J_s in any order k. These definitions, however, then are dependent on the direction x_s, unlike C_k of J.

2. SENSITIVITY TO CORRELATIONS

We call a map in a two-dimensional $\mathbf{x}$ space "disentangled" into two uncorrelated maps if $f(\mathbf{x})$ has the special form

$$f_i^0(x_1, x_2) = f_i^0(x_i), \qquad i = 1, 2 \tag{2.1}$$

i.e., if it is a composition of two independent maps in different subspaces. The invariant measure (1.5) then factorizes with respect to the subspaces:

$$\rho^0(x_1, x_2) = \rho_1(x_1)\, \rho_2(x_2) \tag{2.2}$$

Now let us look for a map that differs only slightly from such a disentangled map,

$$f(\mathbf{x}) = f^0(\mathbf{x}) + \varepsilon g(\mathbf{x}), \qquad |\varepsilon| \ll 1 \tag{2.3}$$

and has the same marginal measures

$$\rho_1(x_1) = \int dx_2\, \rho(x_1, x_2) \tag{2.4}$$

$$\rho_2(x_2) = \int dx_1\, \rho(x_1, x_2) \tag{2.5}$$

In lowest order of ε we can write

$$\rho(\mathbf{x}) = \rho^0(\mathbf{x}) + \varepsilon\sigma(\mathbf{x}) \tag{2.6}$$

with

$$\int dx_1\, \sigma = \int dx_2\, \sigma = 0 \tag{2.7}$$

To separate the influence of correlations upon the cumulants C_k of information loss J from the influence of any other property of a map, we look for the case that J is the same in both maps, at least in lowest order of ε. This means that

$$J(x_1, x_2) = J_1^0(x_1) + J_2^0(x_2) \tag{2.8}$$

holds up to first order of ε; then

$$\int dx_1\, dx_2\, \sigma(J)^k \tag{2.9}$$

vanishes for $k = 1$, but not generally for $k > 1$. Therefore, in the generating function of the cumulants

$$\log\langle e^{\eta J}\rangle = \log \int dx_1\, dx_2\, (\rho^0 + \varepsilon\sigma)\, e^{\eta J} \tag{2.10}$$

the cumulant C_1 has no term that is linear in ε, unlike the cumulants C_k of higher order, $k > 1$.

This means that C_k for $k > 1$, and in particular the variance C_2, are sensitive to correlations between the subspaces in a higher degree than C_1, the mean value $\langle J\rangle$ of information loss.

The proof is valid for any two subspaces x_1, x_2 of a higher dimensional $\mathbf{x}$ space and for dynamical flows as well.

3. THE TRIANGULAR MAP

In the following we shall consider some special one-dimensional maps with real x:

$$x^{(n+1)} = f(x^{(n)}) \tag{3.1}$$

$$J(x) = \log |f'(x)| \tag{3.2}$$

where f' denotes the derivative of f. The mean value of J is here the Liapunov exponent

$$\lambda = \langle \log |f'| \rangle \tag{3.3}$$

We shall discuss the variance

$$V^2 = \langle (\log |f'|)^2 \rangle - \lambda^2 \tag{3.4}$$

First we consider the "triangular map"

$$y^{(n+1)} = g(y^{(n)})$$

$$g(y) = \begin{cases} yc/a & \text{for} \quad y \in (0, a) \qquad (3.6) \\ (1-y)c/a & \text{for} \quad y \in (a, 1) \qquad (3.7) \end{cases}$$

with

$$P = \int_0^a dy\, \rho(y) \leqslant 1 \tag{3.8}$$

where ρ is defined by Eq. (1.13); we obtain

$$\lambda = \log c - P \log a - (1-P) \log(1-a) \tag{3.9}$$

$$V^2 = P(1-P)\{\log[(1-a)/a]\}^2 \tag{3.10}$$

4. THE LOGISTIC MAP

$$f(x) = rx(1-x) \tag{4.1}$$

$$x \in (0, 1), \qquad r \in (0, 4) \tag{4.2}$$

$$J(x) = -\log r - \log(1-2x) \tag{4.3}$$

For the particular value $r = 4$ the map becomes distinguished by the property that the invariant measure $\rho(x)$ can be calculated analytically.[13] The same then is true for λ and V. This is due to the well-known fact that this map is topologically conjugate to the triangular map with $c = 1$, $a = 1/2$. Both maps are connected by the transformation

$$x = (\sin \pi y/2)^2 \tag{4.4}$$

This leads for the logistic map to

$$\rho(x) = \pi^{-1}[x(1-x)]^{-1/2} \tag{4.5}$$

$$\lambda = 2 \int_0^{1/2} dy \log(4 \cos y) = \log 2 \tag{4.6}$$

$$V^2 = 2 \int_0^{1/2} dy\, [\log(4 \cos y)]^2 - \lambda^2 = \pi^2/12 \tag{4.7}$$

(The integrals are listed in Refs. 25–27.)

It is remarkable that V is different from the value zero of V for the conjugate triangular map, whereas λ, as pointed out by Shaw,[22] is the same for both maps. This again demonstrates that in general V describes a new property of a map that is independent of λ.

Plots of the x values from numerical calculations in the whole interval $(0, 4)$ of the parameter r has been given by several authors.[2,10,29]

Numerical calculations of the dependence of λ, V, and $|\lambda|/V$ on the parameter r were performed by Dörpelkus.[30] The value of r was varied in steps of 0.0025. The map was iterated 10^5 times. The results are plotted in Fig. 1. The plots are interpolations between the calculated values.

The interval $(0, 4)$ of r is divided into two essentially different parts separated by a value $r_\infty = 3.5699456$.[2]

For $r < r_\infty$ the system is nonchaotic. Asymptotically, it becomes periodic. For $r > r_\infty$ it is chaotic. The chaos, however, shows different structures for different values r. It shows "windows"[7] with periodic behavior. Inspection of the plots shows that λ and V diverge simultaneously. Such divergences occur in particular at the windows. This can be understood better by first discussing the regime $r < r_\infty$.

The Periodic Regime $(0, r_\infty)$. Here there exists a sequence $1 < r_1 < r_2 < \cdots < r_\infty$ of r values r_n such that in an interval (r_n, r_{n+1}) the map asymptotically tends to a cycle of 2^n values x_i through which the system runs with the period 2^n of i. (In the following we use a subscript on x to distinguish these "branches" of a cycle.) The r_n are bifurcation points in the Feigenbaum diagram. At each r_n a period doubling takes place. In $(0, 1)$ only $x = 0$ is stable. In $(1, r_1)$, only

$$x_0 = 1 - 1/r \tag{4.8}$$

is stable. The two "superstable" cycle branches x_1, x_2 in (r_1, r_2) are solutions of the algebraic equation

$$x = f(x) \tag{4.9}$$

which is of fourth order, but can be solved easily, since the additional two solutions 0, x_0 are known:

$$x_{1,2} = 2^{-1}(1 + r^{-1}) \pm (2r)^{-1}[(r-3)(r+1)]^{1/2} \tag{4.10}$$

Linear fluctuation analysis gives the values $r_1 = 3$ and

$$r_2 = 1 + 6^{1/2} \tag{4.11}$$

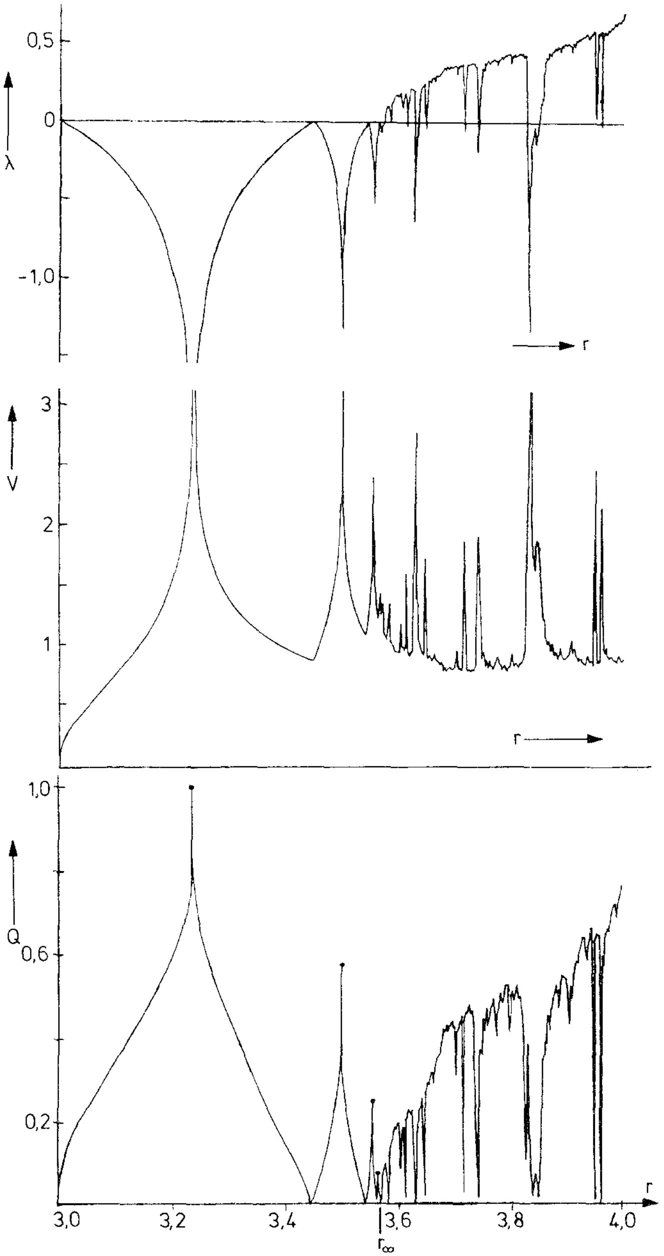

Fig. 1. Plot of λ, V, and Q (from top to bottom) as a function of the parameter r for the logistic map. The plots are interpolations between values calculated for r varied in steps of 0.0025. The map was iterated 100,000 times. The first four cusps of Q belonging to supercycles $r = R_n$ are marked by dots.

as stability limits of x_0 and of $x_{1,2}$. Generally such an analysis gives for any $2^n = m$ cycle the stability condition

$$\prod_{i=1}^{m} |f'(x_i)| \leqslant 1 \tag{4.12}$$

Therefore, in any branching point r_{n+1} the Liapunov exponent

$$\lambda = \sum_i J(x_i) = \sum_i \log |f'(x_i)| \tag{4.13}$$

vanishes, whereas V remains finite. This is why it is more convenient to plot, not the ratio of V over $|\lambda|$, which is the usual measure for relative deviations of J from the average, but the reciprocal

$$Q = |\lambda|/V \tag{4.14}$$

λ becomes $-\infty$ for the so-called "supercycles" (compare, e.g., Ref. 2) with value $r = R_n$, which have a cycle branch $x = 1/2$ with vanishing $f'(x)$. The following always holds:

$$r_n < R < r_{n+1} \tag{4.15}$$

For a supercycle V becomes infinite as well. Yet the ratio Q remains finite. Nevertheless, Q becomes singular as a function of r in a peculiar way. For values r in the neighborhood of R_n we can, for small $|\varepsilon|$, write for the mentioned particular branch

$$x(r) = 1/2 + \varepsilon/R_n \tag{4.16}$$

This means that the deviation of r from R_n is linearized in ε. By restriction to the leading terms in ε one finds with $m = 2^n$:

$$f'(x) = \log |\varepsilon| \tag{4.17}$$

$$\lambda = (\log |\varepsilon| + S)/m \tag{4.18}$$

where S is the sum of $J(x_i)$ of the other branches i of the cycle in R_n. Each branch enters with the same weight $1/m$ into the mean values $\langle J \rangle$ and $\langle J^2 \rangle$.

Therefore we obtain

$$\langle J^2 \rangle = m\langle J \rangle^2 - 2S\langle J \rangle \tag{4.19}$$

$$Q = (m-1)^{-1/2}[1 + Sm(m-1)^{-1}(\log |\varepsilon|)^{-1}] \tag{4.20}$$

This means that the ratio Q has a finite peak ending in $(m-1)^{-1/2}$, but with a logarithmic cusp.

For the 2-cycle in (r_1, r_2) we find in particular

$$(R_1 - 3)(R_1 + 1) = 1, \qquad R_1 = 1 + 5^{1/2} \tag{4.21}$$

which means that $2/R$ is the golden mean.

For $\lambda \to \infty$ the empirical scaling law of Feigenbaum[31,32] holds:

$$(R_n - R_\infty)\,\delta^n \sim 1 \tag{4.22}$$

This gives for the cusp maxima of Q in the limit $n \to \infty$

$$Q \sim 2^{-n/2} \sim (R_n - R_\infty)^{\beta/2} \tag{4.23}$$

$$\beta = \log 2/\log \delta \tag{4.24}$$

where δ is the Feigenbaum constant. It should, however, be stressed that the cusp maxima of Q form a very peculiar subset in the set of all Q.

In conclusion, we note that λ vanishes at the bifurcation points r_n where V remains finite, and that λ and V diverge simultaneously at the points R_n where for one branch of a superstable cycle $x = 1/2$ and the loss becomes $-\infty$. The ratio Q then has a finite cusp.

The Chaotic Regime (r_∞, 4). Here no statements in the same stringent way seem possible. Yet analogies to the periodic regime with respect to the divergences seem helpful. One observes that divergences of λ and V coincide in the chaotic regime as well. They can be seen in the periodic windows. As in the periodic regime, the ratio Q remains finite at these divergence points. The windows are connected with inverse bifurcations. Also in the chaotic regime without windows one observes more or less distinct "branches" as lines of higher density of x points reached by the mapping iteration. If such a line passes the value $x = 1/2$, we can expect a peak of λ and V because for this x value the information loss becomes $-\infty$. The ratio Q would be finite, but with a peak.

The following is remarkable. If we disregard the peaks, there remains a more or less distinct "background" curve for λ, V, and Q, respectively. Whereas the background of λ increases with r, expressing increasing chaos, the background of V, as a measure of the inhomogeneity of the loss flow, remains relatively unchanged in the whole chaotic regime.

5. CONCLUSION

Since the definition of the Liapunov exponent λ and that of the Kolmogorov entropy are based on the mean value of the information loss of a dynamical map or flow, and the mean value can be interpreted as the first cumulant, it seems useful to introduce loss cumulants of higher order as independent characterizations of chaos. It has been shown that the cumulants of higher order are more sensitive to correlations in the dynamics than is the mean value of loss.

The most important of the higher cumulants is the second cumulant, the variance V^2 of the information loss. It has been considered in particular for the logistic map, the standard example of a nonlinear map. Numerical results for λ, V, and the ratio Q defined in Eq. (4.14) are given. Whereas λ and V can diverge for certain values of the map parameter r, Q remains finite. The fact that λ and V diverge simultaneously indicates that the origin of the divergence might be the same in the chaotic as in the periodic regime.

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